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Tachyon condensation of D2/D4-brane system in noncommutative gauge theory

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Abstract: In this paper we construct the 2+1 effective theory of the light states in D2/D4-brane system in the context of noncommutative Yang-Mills theory. This effective theory is noncommutative and tachyonic, however, it is not taking the form of an abelian Higgs model as naively expected. We solve the classical solutions of the effective theory which are nicely corresponding to different states during the tachyon condensation process of the dissolution of D2-brane into D4-brane. We also find that if the expected stable self-dual D0/D4 configuration as the unit-winding vortex exists, it would be highly calibrated in the effective theory and be out of the reach of the analytic solutions.

Keywords: D-branes, Non-Commutative Geometry.

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Contents

1. Introduction

Noncommutative Yang-Mills Theory is extensively studied in a classic paper [1] by Seiberg and Witten in the context of string theory as the effective theory of D-brane world volume with the large B-field on it. In that paper, the D0-brane as a noncommutative instanton on the D4-brane world volume is analyzed, the perturbative string spectrum shows the existence of the towers of light states for large noncommutativity. It has been argued that these large number of light states could be interpretated as the fluctuation modes around the noncommutative instanton. However, there seems no good formalism to describe the dynamics of these light states including tachyon except in the context of noncommutative solitons [2]–[6] as described below. The existence of towers of light states has been shown in other Dp/Dq systems with appropriate large B fields in [7, 8], see also [9].

Recently the unstable solitons (so called fluxons) in $2+1$ noncommutative Yang-Mills theory (NCYM) have been constructed in [4], see also [5, 10, 9, 11, 12]. Surprisingly the fluctuations around the soliton have the same towers of light states spectrum as the ones obtained from open string excitations of D0/D2 in the large noncommutative limit. In a recent paper [13], the light states spectrum of D0/D4 open string with appropriate large B field has been shown to coincide with the fluctuation spectrum of soliton solution in $4+1$ NCYM. In fact, from the discussion in [4, 13], it is easy to generalize this coincidence to D0/D6 and D0/D8 cases (see also [14] for more detailed discussions). The other approach to understand the dynamics of the light states and tachyon is from on-shell open string amplitude (see [15, 16]). It has been shown the low energy effective action of tachyon is a noncommutative Yang-Mills with tachyon covariant coupling. The kinetic term of tachyon is found to be independent of the transverse directions. And there is a damping factor which shows the noncommutative soliton configuration. All of these discussions imply a possible way to understand the large number of light states and the tachyon condensation in the context of NCYM.

In this paper, we discuss the D2/D4-brane system with B field in the framework of NCYM. The effective theory of the Dp/D$(p+2)$-brane system with B-field, especially the quartic tachyon potential has been discussed in the commutative description [15]. However, by taking the advantage of easy construction of the noncommutative solitons which are the co-dimensional D-branes in Sen's tachyon condensation program [17, 18, 19, 3, 4], we reconsider the D2/D4 system in NCYM context and construct the possible solitonic states of the effective theory dictated by the tachyon potential. The reason for the effective field theory to be a reliable description of the tachyon condensation is that the mass of the tachyon is very small and the stringy effect can be suppressed in the large B-field limit.

We will start from the D4-brane worldvolume theory and construct unstable D2-brane as a fluxon from the point of view of the transverse directions. Similar to the D0/D2 fluxon, there exist light fluctuating modes including tachyon which are governed by a 2+1 noncommutative effective theory. One could expect that such an effective action should be reduced to the noncommutative abelian Higgs model with tachyon as a bi-fundamental Higgs field, disregarding the towers of scalars. The tachyon condensation then could result in a noncommutative vortex of nontrivial winding which can be identified as the stable D0-brane on D4-brane world volume as suggested in [4]. Unfortunately this vortex configuration turns out to be highly calibrated and is out of the reach of the analytic solutions in our effective theory description. The main reason for this complication is that the tachyon does not have the canonical bi-fundamental scalar couplings so that the usual partial isometry techniques are inapplicable here.

The paper is organized as follow: in section 2, the results in D0/D2 case are briefly reviewed; in section 3, the effective theory of the light states and the tachyon is given in the case of D2/D4; in section 4, the classical solutions of the effective theory are worked out and the possible existence of the unit-winding vortex solution is discussed; some remarks and conclusion are in section 5. For completeness we give the tachyon spectrum analysis of D0/D4-brane system in the appendix.

2. Light states spectrum of D0/D2-brane system

In this section, we give a brief review of the light states spectrum of D0/D2-brane system to set up the notations for our discussion on D2/D4-branes. The detailed discussions can be found in [4].

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2.1 Soliton solution and effective theory for light states

The action of the two-dimensional noncommutative Yang-Mills theory can be recast into a form of Matrix Model (see also [19].):

$$
\begin{aligned}
S &= -\frac{1}{4g_{YM}^2}\int d^3x\, F_{\mu\nu}F^{\mu\nu} \\
&= \frac{2\pi\theta}{g_{YM}^2}\int dt\,\mathrm{Tr}\left(-\partial_t\overline{C}\partial_t C + \frac{1}{2}\left([C,\overline{C}]+\frac{1}{\theta}\right)^2\right) . \qquad (2.1)
\end{aligned}
$$

Here, following the notation in [4]

$$
\begin{aligned}
C &= a^\dagger - iA_z \\
\overline{C} &= a + iA_{\overline{z}} \qquad (2.2)
\end{aligned}
$$

with $[a,a^\dagger]=1/\theta$.

The equation of motion is

$$\partial_t^2 C = [C,[C,\overline{C}]]\,, \qquad (2.3)$$

and the Gauss law constraint for $A_0=0$ is

$$[\overline{C},\partial_t C]+[C,\partial_t\overline{C}]=0\,. \qquad (2.4)$$

The ground state solution is $C=a^\dagger$ corresponding to a D2-brane. Moreover, there are static soliton solutions corresponding to m D0-branes on the D2-brane worldvolume:

$$C=(S^\dagger)^m a^\dagger S^m + \sum_{i=0}^{m-1} c^i|i\rangle\langle i|\,, \qquad (2.5)$$

where c^i's are the constant moduli parameterizing the locations of the D0-branes, and the "shift" operator S is defined as

$$SS^\dagger = 1\,, \qquad S^\dagger S = 1-P_{m-1}\,, \qquad (2.6)$$

with $P_{m-1}\equiv\sum_{i=0}^{m-1}|i\rangle\langle i|$.

In this paper, we focus on the $m=1$ case. Written in the form of matrix,

$$C=\begin{pmatrix} c^0 & 0 \\ 0 & a^\dagger \end{pmatrix}.$$

For simplicity, we take $c^0=0$.

Given the solution for the soliton configuration of (2.1), we can investigate the fluctuations around the solution. Out of the surprise, the spectrum of the fluctuations agree exactly with the spectrum of light states existing in the D0/D2 system with appropriate large NS B-field[1] [1, 7, 4]. These light states are the collective excitations around the D0-brane soliton, taking the form of

$$\delta C=\begin{pmatrix} A & W \\ \bar{T} & D \end{pmatrix},$$

[1] In fact, there is a minor mismatch: $m^2=1/\theta$ mode is absent in the fluctuation spectrum but this mode exists in the perturbative string spectrum. There is also a mysterious open string scale factor discrepancy between these two spectra [4].

where the field A is the fluctuations of D0-D0 open strings, while the field T and W are D0-D2 excitations, and the field D is D2-D2 excitations which should be integrated out to get the effective potential for the tachyon describing the dissolution of D0-brane. In components,

$$T = \sum_{k=0} T_k|0\rangle\langle k+1|\,, \qquad W = \sum_{k=0} W_k|0\rangle\langle k+1|\,, \qquad A = A_0|0\rangle\langle 0|\,. \tag{2.7}$$

Note that T_0 mode is tachyonic.

In evaluating the effective potential

$$V = \frac{1}{2}\left([C,\overline{C}] + \frac{1}{\theta}\right)^2 \tag{2.8}$$

we will restricted only to the sector

$$D = \sum_{k=0} D_{k+1,k}|k+1\rangle\langle k|\,, \tag{2.9}$$

for the reasons as pointed out in [4]. After some lengthy calculation, the effective potential takes the form of the sum of complete squares

$$\begin{aligned} 2V \;=\; & M_{-1}^2 + M_0^2 + (N_0\overline{N}_0 + \text{c.c.}) + \sum_{k=1} M_k^2 + \sum_{k=1}(N_k\overline{N}_k + \text{c.c.}) + \sum_{k=1}(Q_k\overline{Q}_k + \text{c.c.}) + \\ & + \sum_{i=1}\sum_{k=1}(R_{i,k}\overline{R}_{i,k} + \text{c.c.})\,, \end{aligned} \tag{2.10}$$

where

$$M_k = E_k\overline{E}_k - \overline{E}_{k+1}E_{k+1} + \frac{1}{\theta} - \frac{k+2}{2k+3}\overline{J}_{k+1}J_{k+1} + \frac{k-1}{2k-1}\overline{J}_{k-1}J_{k-1}\,, \tag{2.11}$$

$$N_k = \sqrt{\frac{1}{2k+1}}J_k(\sqrt{k+1}\overline{E}_k + \sqrt{k}\overline{E}_{k+1}) - \left(\sqrt{\frac{k+2}{2k+3}}\overline{A}J_{k+1} + \sqrt{\frac{k-1}{2k-1}}AJ_{k-1}\right), \tag{2.12}$$

$$M_{-1} = -T_0\overline{T}_0 + \frac{1}{\theta} + [A_0,\overline{A}_0] + \sum_{k=1}\frac{1}{2k+1}J_k\overline{J}_k\,, \tag{2.13}$$

$$M_0 = \overline{T}_0T_0 - E_1\overline{E}_1 + \frac{1}{\theta} - \frac{2}{3}\overline{J}_1J_1\,, \tag{2.14}$$

$$N_0 = A_0T_0 - \sqrt{\frac{2}{3}}\overline{A}_0J_1\,, \tag{2.15}$$

$$Q_k = -\Big(\sqrt{\frac{2(k+2)}{3(2k+3)}}\overline{J}_1J_{k+1} + \sqrt{\frac{k-1}{2k-1}}\overline{T}_0J_{k-1}\Big)\,, \tag{2.16}$$

$$R_{i,k} = -\sqrt{\frac{(i+2)(i+k+2)}{(2i+3)(2i+2k+3)}}\overline{J}_{i+1}J_{i+k+1} - \sqrt{\frac{(i-1)(i+k-1)}{(2i-1)(2i+2k-1)}}\overline{J}_{i-1}J_{i+k-1}\,, \tag{2.17}$$

and E_k's are defined as

$$E_k \equiv D_{k,k-1} + \sqrt{\frac{k}{\theta}}\,. \tag{2.18}$$

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In the above we have used the Gauss Law constraint

$$0 = \frac{1}{\sqrt{2k+1}}\left(\sqrt{k}W_k + \sqrt{k+1}T_{k+1}\right), \tag{2.19}$$

and define a new scalar

$$J_k = \frac{1}{\sqrt{2k+1}}\left(\sqrt{k+1}W_{k-1} - \sqrt{k}T_{k+1}\right), \tag{2.20}$$

so that

$$W_{k-1} = \sqrt{\frac{k+1}{2k+1}}J_k\,, \qquad T_{k+1} = -\sqrt{\frac{k}{2k+1}}J_k \tag{2.21}$$

with $T_1 = J_0 = 0$ and $k = 1, 2, 3, \ldots$

2.2 The spectrum and tachyon condensation

Some remarks are in order about the above results. Firstly, In the case of D0/D2, the fluctuations around unstable soliton configuration are just complex excitations, and they can be described by quantum mechanics. We will show in the following section in the case of D2/D4, the fluctuations have to be governed by 2+1 dimensional noncommutative field theory. We have retained the order of the fields in (2.10) so that the result can be directly carried to its noncommutative version for the D2/D4-brane system.

Another interesting point is that what is the fate of the fluctuations after tachyon condensation. As discussed in [4], there exist two vacua: one is unstable vacuum for the undissolved D0/D2, and the other is the stable vacuum for the dissolved D0 in D2. Thanks to the sum of complete squares of (2.10), the stable vacuum is at the minimum $V = 0$ with

$$M_{-1} = M_0 = M_k = N_0 = N_k = Q_k = R_{i,k} = 0\,, \tag{2.22}$$

which can be solved by

$$T_0 = \sqrt{\frac{1}{\theta}}\,, \qquad E_k = \sqrt{\frac{k+1}{\theta}}\,, \tag{2.23}$$

and the other fields are set to zero. On the other hand, the unstable vacuum of $D0/D2$-brane is at

$$T = W = D = A = 0\,, \tag{2.24}$$

with $2V = 1/\theta^2$.

It's easy to read out the mass spectrum of excitations from the quadratic terms in (2.10): Around the unstable vacuum, the worldvolume gauge field A_0 of D0-brane is massless, while the tachyon T_0 has the mass-square $-1/\theta$, and the complex massive scalar J_k's has the mass-square $(2k+1)/\theta$, which are light for large θ and are exactly the masses of light states of D0-D2 open string in the zero slope limit [7, 1]. Around the stable vacuum, the excitations A, J_k get extra masses from the tachyon condensation. In fact, the relevant quadratic terms of the action turns out to be[2]

$$\frac{1}{\theta}A_0\overline{A}_0 + \frac{1}{\theta}\sum_{k=1}\frac{(k+1+\sqrt{k(k+2)})^2}{2k+1}J_k\overline{J}_k\,. \tag{2.25}$$

[2]We neglect the mass terms for $D_{k,k-1}$ scalars since they are the D2-D2 open string fluctuations which are not localized around the D0-brane, and should be integrated out for the effective theory.

Obviously the excitation A_0 get mass implying the dissolution of the D0-brane is described by Higgs mechanism, and J_k get a little bigger mass damping the fluctuations after tachyon condensation. This tells that the collective fluctuations do not decouple but are damped after the tachyon condensation.

3. Effective theory of light states of D2/D4-brane system

In this section, we would like to discuss the effective noncommutative field theory of light states in D2/D4-brane system. We will start from the D4-brane world volume action and construct the D2-branes as the unstable soliton. This is quite similar to the construction of D0/D2. The key difference is that we will obtain an effective noncommutative field theory which governs the dynamics of the fluctuations.

3.1 The world volume theory of a D4-brane

The world volume action of a D4-brane with a B field background on it is a $(4+1)$-dimensional noncommutative Yang-Mills theory, for simplicity B takes a canonical form, i.e.

$$B_{\mu\nu} = \begin{pmatrix} 0 & B_{12} & & \\ -B_{12} & 0 & & \\ & & 0 & B_{34} \\ & & -B_{34} & 0 \end{pmatrix}. \tag{3.1}$$

As usual, $B_{12}(B_{34})$ induces noncommutativity in $1,2(3,4)$-directions with parameter $\theta_1(\theta_2)$.

Then the action can be written in the form

$$\begin{aligned} S &= \frac{-1}{4g_{YM}^2}\int d^5x\; F_{\mu\nu}F^{\mu\nu} & (3.2)\\ &= -\frac{4\pi^2\theta_1\theta_2}{g_{YM}^2}\int dt\,\mathrm{Tr}\Bigg(-\partial_t\overline{Y}\partial_t Y - \partial_t\overline{Z}\partial_t Z + \frac{1}{2}\left([Y,\overline{Y}] + \frac{1}{\theta_1}\right)^2 + \\ &\qquad + \frac{1}{2}\left([Z,\overline{Z}] + \frac{1}{\theta_2}\right)^2 + [\overline{Y},\overline{Z}][Z,Y] + [Y,\overline{Z}][Z,\overline{Y}]\Bigg)\,, & (3.3) \end{aligned}$$

where

$$Y = a_y^\dagger - iA_y(y,\overline{y},z,\overline{z})\,, \qquad Z = a_z^\dagger - iA_z(y,\overline{y},z,\overline{z}) \tag{3.4}$$

with $[a_{y,z}, a_{y,z}^\dagger] = 1/\theta_{1,2}$. As be will be shown that the cross terms between Y and Z will give the correct kinetic terms of the collective excitations in the effective theory of D2/D4 system.

We are only interested in the classical static solution, so the equations of motion are

$$[Y,[Y,\overline{Y}]] + [\overline{Z},[Y,\overline{Z}]] + [Z,[Y,\overline{Z}]] = 0\,, \tag{3.5}$$
$$[Z,[Z,\overline{Z}]] + [[Y,Z],\overline{Y}]] + [[\overline{Y},Z],\overline{Y}] = 0\,, \tag{3.6}$$

and the Gauss Law constraints are just the same as (2.4).

The ground state(zero energy) solution corresponding a D4-brane is

$$Y = a_y^\dagger \otimes I_z \,, \qquad Z = I_y \otimes a_z^\dagger \,, \tag{3.7}$$

where $I_{y(z)}$ is the identity operator in the subspace of $1,2(3,4)$ noncommutative plane, which indicates the infinite extension along the corresponding directions.

3.2 Effective noncommutative field theory

The soliton solution of (3.2) corresponding to a D2/D4-brane configuration is

$$Y = C_y \otimes P_{0z} \,, \qquad Z = I_y \otimes S_z^\dagger a_z^\dagger S_z \,. \tag{3.8}$$

Here C_y can be thought as the background field coupled to the fluctuating modes. Note that the solution corresponding to the D2/D4 with zero background $C_y = 0$ has the zero-point energy

$$E_0 = \frac{2\pi^2}{g_{YM}^2}\left(\frac{\theta_1}{\theta_2} + \frac{\theta_2}{\theta_1}\right) \mathrm{Tr}_y\, I_y = E_{D0}\, \mathrm{Tr}_y\, I_y \,, \tag{3.9}$$

the trace Tr_y is taken over $1,2$-noncommutative plane, and E_{D0} is the energy of a D0-brane on D4-brane as given in the appendix. The form of the zero-point energy indicates the underlying D2-brane which can be viewed as the bound state of infinite D0-branes on the $1,2$-plane.

From the D4-brane point of view, the solution (3.8) gives an unstable soliton which is just a D2-brane extending along the $1,2$-directions but localizing in the $3,4$-directions. Therefore, we can obtain a noncommutative effective field theory for the collective excitations living on the 1,2-plane by treating the solution (3.8) as a D0/D2-brane system in the $3,4$-directions. The collective excitations as the fluctuations around the configuration (3.8) can be parameterized as

$$\delta Z = \begin{pmatrix} A(y,\overline{y}) & W(y,\overline{y}) \\ \overline{T}(y,\overline{y}) & D(y,\overline{y}) \end{pmatrix} . \tag{3.10}$$

Note that all fluctuations now are the fields depending on the $1,2$ coordinates $y, \overline{y}$ and their products should be understood as star-products. Similar to the D0/D2 system, A corresponds to the 2-2 string excitations, T and W corresponds to the 2-4 string excitations and D to the 4-4 string excitations, which should be integrated out in the effective theory. Again all the modes becomes light states of the string theory in the large noncommutativity limit [1, 7].

To obtain the effective theory, we need to expand A, W, T, D into components as done in (2.7) and (2.9) but now each component field is function of y and $\overline{y}$. Again we keep on the branch of D as given by (2.9).

First the term $(1/2)\,\mathrm{Tr}([Z,\overline{Z}] + 1/\theta_2)^2$ in (3.3) gives part of the effective action S_1 for the fluctuation fields as following

$$\begin{aligned} 2S_1 = \mathrm{Tr}_y \Bigg(& \frac{1}{\theta_2^2} + \ M_{-1}^2 + M_0^2 + (N_0\overline{N}_0 + \text{c.c.}) + \sum_{k=1} M_k^2 + \sum_{k=1} (N_k\overline{N}_k + \text{c.c.}) + \\ & + \sum_{k=1} (Q_k\overline{Q}_k + \text{c.c.}) + \sum_{i=1}\sum_{k=1} (R_{i,k}\overline{R}_{i,k} + \text{c.c.}) \Bigg) , \end{aligned} \tag{3.11}$$

where the fields M, N, Q, R are the same as the ones defined in (2.11) to (2.17) with θ replaced by θ_2. As remarked before, although these fields are noncommuting, we just directly carry (2.10) of D0/D2 system to (3.11) because we have retained the order of the fields in the expression of (2.10).

On the other hand, the rest of the action (3.3) will give another part of the effective action with background field couplings

$$
\begin{aligned}
S_2 = &\ \mathrm{Tr}_y \left(\frac{1}{2} \left([C_y, \overline{C}_y] + \frac{1}{\theta_1} \right)^2 \right) + \\
&+ \mathrm{Tr}_y\, \mathrm{Tr}_z ([\overline{C}_y \otimes P_{0z}, \delta\overline{Z}][\delta Z, C_y \otimes P_{0z}] + [C_y \otimes P_{0z}, \delta\overline{Z}][\delta Z, \overline{C}_y \otimes P_{0z}])\,. \qquad (3.12)
\end{aligned}
$$

Here we have suppressed the fluctuation δY which may be relevant for the complete light states spectrum as discussed in the appendix for the D0/D4-brane system.

The total energy for the effective theory of the light states is given by

$$
E = \frac{4\pi^2\theta_1\theta_2}{g_{YM}^2}(S_1 + S_2)\,. \qquad (3.13)
$$

4. Tachyon condensation of D2/D4-brane system

In this section we will discuss the tachyon condensation of the unstable D2-brane from the effective field theory we obtained in the last section.

4.1 Tachyon condensation in an abelian Higgs model?

From the 2+1 effective field theory action S_1+S_2 the equations of motion can be obtained as usual. The tachyon field and the scalar fields coming from the 2-4 string are bi-fundamental complex scalars, one would expect that the effective action is an abelian Higgs model which has the noncommutative vortex solution of nontrivial winding [20, 21, 10] which could be identified as a D0-brane. It turns out this is not the case since their explicit forms of couplings with gauge fields are not of the form of bi-fundamental or fundamental scalar coupling if we keep the potential in the Higgs form.

To be more explicit, we have two gauge fields: one is the gauge field A_0 from 2-2 string which reflects the dynamics of the underlying D2-brane, and the other is the background gauge field C_y which represents the more general D2/D4-brane configurations and is turned off in the following discussions. For comparison, let us recall the form of the covariant coupling of a bi-fundamental scalar ϕ

$$
\mathrm{Tr}\left(\overline{C}_1\phi\overline{\phi}C_1 + C_1\phi\overline{\phi}\overline{C}_1 + \overline{C}_2\phi\overline{\phi}C_2 + C_2\phi\overline{\phi}\overline{C}_2 - 2C_1\phi\overline{C}_2\overline{\phi} - 2C_2\overline{\phi}\overline{C}_1\phi\right). \qquad (4.1)
$$

with $C_1 = a^\dagger - iA_1$ and $C_2 = a^\dagger - iA_2$.

On the other hand, the relevant action for tachyon condensation from S_1+S_2 by setting $J_k=0$ is

$$S_{\rm eff} = {\rm Tr}_y \left(\frac{1}{2}\left([C_y,\overline{C}_y]+\frac{1}{\theta_1}\right)^2 + \frac{1}{2}\left(-T_0\overline{T}_0+\frac{1}{\theta_2}+[A_0,\overline{A}_0]\right)^2 + A_0T_0\overline{T}_0\overline{A}_0 + \right.$$
$$\left. + C_yT_0\overline{T}_0\overline{C}_y + \overline{T}_0C_y\overline{C}_yT_0 + +[\overline{C}_y,\overline{A_0}][A_0,C_y] + [C_y,\overline{A_0}][A_0,\overline{C}_y] \right) . \quad (4.2)$$

Obviously the gauge fields C_y and A_0 have the correct kinetic terms; the tachyon T has the usual Higgs potential but its kinetic term is not in the canonical form of (4.1). Therefore the effective theory (4.2) is not an abelian Higgs model as naively expected.

4.2 The classical solutions and tachyon condensation

In the following we will solve the classical solutions of the effective theory which are relevant to the tachyon condensation of the D2/D4-brane system.

From the effective action (4.2) we derive the equations of motion for T_0, A_0 and C_y respectively:

$$0 = \left(C_y\overline{C}_y + \overline{C}_yC_y - T_0\overline{T}_0 + \frac{1}{\theta_2} - [A_0,\overline{A}_0] + \overline{A}_0A_0\right)T_0 \,, \quad (4.3)$$
$$0 = [[A_0,C_y],\overline{C}_y] + [[A_0,\overline{C}_y],C_y] - [-T_0\overline{T}_0 + [A_0,\overline{A}_0],A_0] - A_0T_0\overline{T}_0 \,, \quad (4.4)$$
$$0 = [[C_y,\overline{C}_y],C_y] + [\overline{A}_0,[A_0,C_y]] + [[C_y,\overline{A}_0],A_0] + C_yT_0\overline{T}_0 + T_0\overline{T}_0C_y \,. \quad (4.5)$$

There are many solutions of the equations of motion by simple inspection and by using the solution generating method based on partial isometry [10]. We will discuss the relevant ones which can be interpreted as the initial, intermediate and final states of the tachyon condensation. We also discuss the solutions as the possible candidates for D0/D4, however, it turns out that none of them matches the energy spectrum of D0/D4 given in the appendix.

The first interesting configuration is the stable vacuum known as the "nothing state" representing the complete dissolution of the D2-brane fluxon into the D4-brane. By definition all the gauge fields should be set to zero and tachyon vev reach the bottom of its potential, thus

$$C_y = 0 \,, \qquad T_0 = \frac{1}{\sqrt{\theta_2}} \,, \qquad A_0 = 0 \,, \quad (4.6)$$

which indeed solves the equations of motion. Unlike the usual stable ground state of tachyon condensation, this configuration has the energy $E_{\rm nothing} = ((2\pi^2\theta_2)/(g_{YM}^2\theta_1))\,{\rm Tr}_y\, I_y$ which should be subtracted out from the energy of the other configurations. Note that $E_{\rm nothing} < E_0$ of (3.9) as a consistency check, also that $E_{\rm nothing}$ is not symmetric with respect to the exchange of $\theta_{1,2}$ which can be understood as the tachyon condensation relic of the localized D2-brane breaking the rotational symmetry.

By using the solution generating method of the partial isometry, one can generate new solutions with respect to the above solutions. For example, from the nothing state, we will have the new solution

$$C_y = 0\,, \qquad T_0 = \frac{1}{\sqrt{\theta_2}}(1 - P_{0y})\,, \qquad A_0 = \alpha_0 P_{0y}\,, \tag{4.7}$$

where α_0 is an arbitrary constant. This solution has the α_0-independent finite net energy $\Delta E = (2\pi^2\theta_1)/(g_{YM}^2\theta_2)$ with respect to the $E_{\rm nothing}$ of the nothing state. This configuration represents the dissolution of the D2-brane into a Gaussian bump carrying no magnetic flux because $[A_0, \overline{A}_0] = 0$. Moreover, it has a tachyonic mode with mass-squared $-1/\theta_2$ indicating the tendency of the complete dissolution of the D2-brane. This is in contrast to the D0/D4-brane configuration as analyzed in the appendix, which carries flux or instanton charge and is stable if $\theta_1 = \theta_2$.

In principle we would think the corresponding D0/D4 configuration as given in the appendix will appear near the tachyon vacuum such that the tachyon vev will take the form as in (4.6) or (4.7). One may then wonder if there exist a modified solution of (4.6) or (4.7) with nontrivial A_0 configuration such that one can identify it as the D0/D4 configuration with the correct net energy E_{D0}?

4.2.1 Vortex solution of unit winding?

One of the possible candidates for the $D0/D4$ as suggested in [4] and [21], also in the appendix is the vortex solution of unit-winding which takes the following ansatz form

$$T_0 = S_y \equiv \sum_{k=0} |k\rangle\langle k+1| \quad \text{or} \quad S_y^\dagger \qquad C_y = \beta S_y^\dagger a_y^\dagger S_y; \qquad A_0 = \alpha S_y^\dagger a_y^\dagger S_y\,, \tag{4.8}$$

where $|\alpha|^2 + |\beta|^2 = 1$ from the requirement of the unit winding. If our tachyon is a bi-fundamental scalar in the abelian-Higgs model, then (4.8) will solve the equations of motion, and the vortex solution can be interpreted as the D0/D4. However, this is not the solution since our effective theory is not a bi-fundamental abelian-Higgs model.

We should then try more general ansatz for the tachyon as

$$T_0 = \sum_{k=0} t_k |k\rangle\langle k+1| \qquad \text{or} \qquad \sum_{k=0} w_k |k+1\rangle\langle k|\,. \tag{4.9}$$

Note that if T_0 contain both the above sectors, then there is no consistent truncation to these subspace while solving the equations of motion. For the moment we still require the simple form of the gauge fields as in (4.8) such that all the double commutators in (4.3)–(4.5) involving only the gauge fields will vanish.

Using the ansatz, the equation (4.3) gives

$$|t_{k-1}|^2 = \frac{1}{\theta_2} + \frac{1}{\theta_1}(2k - 1 + |\alpha|^2(2-k))\,, \qquad k = 1, 2, 3, \ldots \tag{4.10}$$

Similarly for the case of w_k. However, the equation (4.5) gives

$$|t_k|^2 + |t_{k-1}|^2 = 0\,, \qquad k = 1, 2, 3, \ldots \tag{4.11}$$

It is clear that the two conditions contradict so that there is no solution for this ansatz.

The most generic ansatz is to also assume the general form for the gauge fields as

$$C_y = \frac{1}{\sqrt{\theta_1}} \sum_{k=0} c_k |k+1\rangle\langle k| \,, \qquad A_0 = \frac{1}{\sqrt{\theta_1}} \sum_{k=0} a_k |k+1\rangle\langle k| \,, \tag{4.12}$$

and the unit winding requirement

$$\left\{ \left([A_0, \bar{A}_0] + \frac{1}{\theta_1} \right) + \left([C_y, \bar{C}_y] + \frac{1}{\theta_1} \right) \right\} = \frac{1}{\theta_1} |0\rangle\langle 0| \tag{4.13}$$

will constrain a_ks and c_ks by

$$|a_0|^2 + |c_0|^2 = 1 \,, \tag{4.14}$$

$$|a_k|^2 + |c_k|^2 = |a_{k-1}|^2 + |c_{k-1}|^2 + 2 \,, \qquad k = 1, 2, 3, \ldots \tag{4.15}$$

In (4.13) we have assumed the unit flux is completely localized in the $|0\rangle\langle 0|$ sector, more general situation with the unit flux scattered in different sectors will lead to more complicated constraints on a_k and c_k.

The ansatz (4.12) makes the double commutators in the equations of motion non-vanishing and leads to the very complicated coupled *cubic* recursive equations for a_k, c_k and c_k's so that one cannot obtain the analytic solutions. However, this does not rule out the possible existence of a highly calibrated unit-winding solution which is just out of the reach of the analytic solutions.

4.2.2 The other non-D0/D4 solutions

It is also interesting to study the configurations near the "top" of the tachyon potential. The configuration right on the top of the potential is

$$C_y = 0 \,, \qquad T_0 = 0 \,, \qquad A_0 = a_y^\dagger \tag{4.16}$$

which also solves the equations of motion. Note that the tachyon vev is zero and the gauge field A_0 representing the D2-brane is turned on.[3] This configuration has the energy $E = E_{\text{nothing}} + (2\pi^2/g_{YM}^2)(\sqrt{\theta_1/\theta_2} - \sqrt{\theta_2/\theta_1})^2 \ \mathrm{Tr}_y\, I_y$ which is always larger than the energy E_{nothing} as expected unless $\theta_1 = \theta_2$. Note that the energy bears similar form of the one for D0/D4-brane system [4] except that there is an Tr_y factor indicating the infinite number of the constituent D0-branes.

To check the stability of (4.16) we need to look into its fluctuation spectrum which takes the following form

$$\left(\frac{1}{\theta_2} - \frac{1}{\theta_1} \right) \mathrm{Tr}_y (-\delta T_0 \delta \overline{T}_0 + [a_y^+, \delta A_0][\delta \overline{A}_0, a_y]) + a_y^+ \delta T_0 \delta \overline{T}_0 a_y \,. \tag{4.17}$$

The lowest mode of δT_0 has the mass-square $2/\theta_1 - 1/\theta_2$ and the modes of δA_0 has the mass-square proportional to $1/\theta_2 - 1/\theta_1$ so that there is a window $2\theta_2 \geq \theta_1 \geq \theta_2$ for the

[3] For more general consideration, one can also turns on the background gauge field C_y in (4.16) such that $C_y = a^\dagger$. This kind of configurations have the energy $E = (2\pi^2/g_{YM}^2)(2\theta_2/\theta_1 + (\sqrt{\theta_2/\theta_1} - \sqrt{\theta_1/\theta_2})^2)\, \mathrm{Tr}_y\, I_y$.

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absence of the tachyon. There is no clear physical reason to explain the existence of such a window which implies the possibility of stable configurations. Outside this window, the configuration is unstable and will condense toward the nothing state.

Finally there is a one-flux solution near the top of the potential which however can not be generated from (4.16) by the partial isometry, taking the form

$$C_y = 0\,, \qquad T_0 = \frac{1}{\sqrt{\theta_2}} P_{0y}\,, \qquad A_0 = S_y^\dagger a_y^\dagger S_y \tag{4.18}$$

with the negative net energy $\delta E = -(2\pi^2/g_{YM}^2)(\sqrt{\theta_1/\theta_2} - \sqrt{\theta_2/\theta_1})^2$ with respect to the energy of the configuration (4.16) indicating that a Gaussian flux bump is created on D2/D4 due to the tachyon condensation. This flux bump is nothing but a hole left by an evacuated D0-brane out of D2-brane since it has -1 flux with respect to the flux number of the configuration (4.16). Moreover, this configuration is unstable as one can see from its fluctuation spectrum so that the system will continue to condense and lower its energy until it reaches the nothing state.

The other solutions with the similar kind of properties of the above flux bump are listed below:

$$\begin{array}{llll}
1)\ C_y = a_y^\dagger\,, & T_0 = 0\,, & A_0 = 0\,, & \\
2)\ C_y = a_y^\dagger\,, & T_0 = 0\,, & A_0 = \kappa a_y^\dagger\,, & \\
\quad\text{where } \kappa \text{ is a constant.} & & & \\
3)\ C_y = S_y^\dagger a_y^\dagger S_y\,, & T_0 = 0\,, & A_0 = 0\,, & \\
4)\ C_y = S_y^\dagger a_y^\dagger S_y\,, & T_0 = 1/\sqrt{\theta_2} P_{0y}\,, & A_0 = 0\,, & \\
5)\ C_y = S_y^\dagger a_y^\dagger S_y\,, & T_0 = 0\,, & A_0 = S_y^\dagger a_y^\dagger S_y\,, & \\
6)\ C_y = S_y^\dagger a_y^\dagger S_y\,, & T_0 = 1/\sqrt{\theta_2} P_{0y}\,, & A_0 = S_y^\dagger a_y^\dagger S_y\,. &
\end{array} \tag{4.19}$$

5. Conclusion

Starting from the 4+1 NCYM we have constructed the noncommutative effective theory for the light states of D2/D4-brane system. Before examining the details of the effective theory, it is quite tempting to expect a noncommutative abelian Higgs model such that its vortex solution may be interpreted as some stable D0/D4 configurations given in the appendix. Unfortunately, this is not the story we find above. In fact, the effective action is not an abelian Higgs model so that if the unit-winding D0/D4 solution exists it would be out of the reach of the analytic solutions. Despite that, some of the solutions of the effective theory have nice interpretations in the process of tachyon condensation describing the complete dissolution of D2 into D4.

The similar situation happens in the Dp-anti-Dp-brane system where it is difficult to construct an effective abelian Higgs theory and also a tachyonic funnel solution as the noncommutative vortex in the context of NCYM at the large B-field limit [22]. This may be seen as the limitations of the NCYM as the effective theory in describing some configurations made of highly calibrated collective modes in the tachyon condensation. This also indicates the need of carefully choosing the variables of the effective theory to describe some nontrivial configurations; a famous example is how to describe closed string around the tachyon vacuum [23]. A direct study of the tachyon condensation of these systems in the off-shell string theory should be pursued to understand the complications due to the towers of both light and stringy states which are completely neglected in our current treatment.

A final side remark is about some novel solutions of the equations of motion (3.5) and (3.6), which has the following form similar to D0/D4 configuration given in the appendix:

$$\begin{aligned} Y &= S_y^{\dagger m} a_y^\dagger S_y^m \otimes P_{kz} \\ Z &= P_{ly} \otimes S_z^{\dagger n} a_z^\dagger S_z^n \,, \end{aligned} \tag{5.1}$$

where $k \leq n$ and $l \leq m$. In fact, here the Y and Z decouple completely. The energy of such solutions is

$$E = \frac{2\pi^2}{g_{YM}^2}\left(\frac{\theta_2}{\theta_1}mk + \frac{\theta_1}{\theta_2}nl\right)$$

which is finite. The interpretation of these solutions in string theory is unclear at this moment. Some of them could be the solutions $T^{\dagger n} a_m^\dagger T^n$ as in [4] and be thought as the intersecting localized D2-branes as considered in [24]. It deserves more study of these configurations.

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A. The light states spectrum of D0/D4-brane

The soliton solution of (3.2) corresponding to a D0/D4-brane is given by

$$Y = S_y^\dagger a_y^\dagger S_y \otimes P_{0z}\,, \qquad Z = P_{0y} \otimes S_z^\dagger a_z^\dagger S_z\,, \tag{A.1}$$

where $S_{y(z)}$ is the shift operators in the $1,2(3,4)$-directions as defined in (2.6), and $P_{0y(z)}$ is the corresponding zero-sector projection operator. The energy of this configuration is finite and equals to $E_{D0} \equiv (2\pi^2/g_{YM}^2)(\theta_1/\theta 2 + \theta_2/\theta_1)$. In [4] it has been conjectured that the fluctuation spectrum contains a tachyon unless the B field configuration is (anti-)self-dual, i.e. $\theta_1 = \pm\theta_2$. This fact is also shown in the perturbative string theory calculations as a BPS condition for preserving the supersymmetry [1, 7]. In the following we will demonstrate this explicitly in the context of NCYM.

The complete fluctuation spectrum is quite complicated, we are only interested in the A_0 and T_0 sectors which correspond to the gauge field and tachyon. Then the fluctuation can be written in the following form

$$\delta Y = (A_{0y}|0\rangle\langle 0| + \overline{T}_{0y}|1\rangle\langle 0|) \otimes P\,, \qquad \delta Z = P \otimes (A_{0z}|0\rangle\langle 0| + \overline{T}_{0z}|1\rangle\langle 0|)\,, \tag{A.2}$$

where $P = (|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0|)$ is the operator representing the most possible fluctuation along the transverse directions caused by the A_0, T_0 sector due to the energy conservation. We also assume that we have fluctuations around an almost self-dual configuration so that we choose the same P to both δY and δZ to maintain the rotational symmetry of the configuration.

After some calculations by using the above ansatz, the quadratic action for A_0 and T_0 is

$$\left(\frac{1}{\theta_2} - \frac{1}{\theta_1}\right) |T_{0y}|^2 + \left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right) |T_{0z}|^2 + \frac{2}{\theta_2}|A_{0y}|^2 + \frac{2}{\theta_1}|A_{0z}|^2 \,, \tag{A.3}$$

where the negative contribution to the tachyon mass comes from the same origin as in the D0/D2 case, and the positive one from the terms of the commutators between Y and Z.

It is obvious that there is always a tachyonic mode unless the configuration is self-dual, i.e. $\theta_1 = \theta_2$. However, the gauge fields become massive, implying no massless modes in the fluctuation spectrum. In [13, 14], the full spectrums of the scalar fluctuations and the gauge field fluctuations have been worked out by using the other basis.

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